Problem proposed by Arkady Alt, San Jose, California, USA.
Upper bound for product of two medians.

Let m_a, m_b be medians of a triangle with sidelengths a, b, c. Prove that

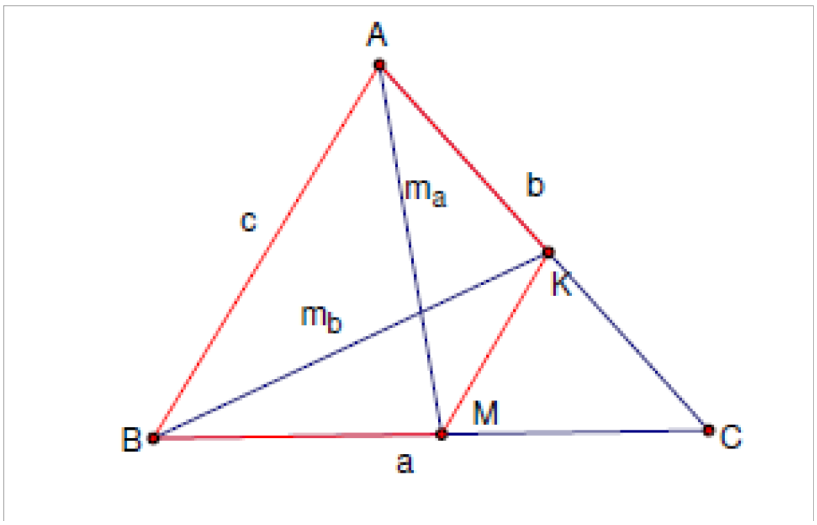
$$m_a m_b \leq \frac{2c^2 + ab}{4}.$$

Solution 1.

Since $m_a^2 = \dfrac{2(b^2+c^2) - a^2}{4}, m_b^2 = \dfrac{2(c^2+a^2) - b^2}{4}$ then

$$16\left(\left(\frac{2c^2+ab}{4}\right)^2 - m_a^2 m_b^2\right) = (2(b^2+c^2) - a^2)(2(c^2+a^2) - b^2) -$$
$$(2c^2+ab)^2 = 2\left((a^2-b^2)^2 - c^2(a-b)^2\right) =$$
$$2(a-b)^2(a+b+c)(a+b-c) \geq 0.$$

Solution 2.



Applying Ptolemy's Inequality to quadrilateral (trapezoid) $AKMB$ we obtain

$$BK \cdot AM \leq AB \cdot MK + AK \cdot BM \Leftrightarrow m_a m_b \leq c \cdot \frac{c}{2} + \frac{b}{2} \cdot \frac{a}{2} \Leftrightarrow m_a m_b \leq \frac{2c^2+ab}{4}.$$